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THE FUNDAMENTAL FREQUENCY OF TRANSVERSE VIBRATIONS OF RECTANGULARLY ORTHOTROPIC, CIRCULAR, ANNULAR PLATES WITH SEVERAL COMBINATIONS OF BOUNDARY CONDITIONS

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1. INTRODUCTION

Solutions of the differential systems governing the problems of transverse vibrations of isotropic, circular, annular plates have been known for over half a century.† Admittedly some of the results available do not possess sufficient accuracy as shown in recent investigations [2–3]. In the case of rectangularly orthotropic, circular, annular plates the studies on transverse vibrations are considerately more modern and, apparently, the first effort was reported in reference [4]. Recent studies on vibrating, rectangularly orthotropic annular plates with a free inner edge have recently appeared [5–7]. Simple polynomial approximations yield good engineering accuracy for geometric configurations characterized by (inner radius)/(outer radius) $\leqslant 0{\cdot}70$, at least when compared with numerical predictions achieved using a very accurate finite element code [8].

The present paper deals with the determination of the fundamental frequency of transverse vibration of the rectangular orthotropic, circular annular plates depicted in Figure 1. Four different boundary arrangements are considered: Case A, simple supported edges; Case B, outer boundary simply supported and clamped inner edge; Case C, clamped edges; Case D, clamped at the outer edge and simply supported inner edge.

2. APPROXIMATE ANALYTICAL SOLUTION

By making use of Lekhnitskii's classical notation [9] one expresses the governing functional in the form

$$J(W)=\tfrac{1}{2}\int\int\left[D_1\left(\frac{\partial^2 W}{\partial x^2}\right)^2+2D_1\mu_2\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+D_2\left(\frac{\partial^2 W}{\partial y^2}\right)^2+4D_k\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y-\frac{\rho\omega^2}{2}\int\int hW^2\,\mathrm{d}x\,\mathrm{d}y. \qquad (1)$$

† Leissa's classical treatise contains exhaustive numerical information on the subject matter [1].

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The displacement amplitude W will be expressed in polar coordinates since they are natural to the boundaries of the structural element under study. In view of this, the partial derivatives appearing in equation (1) will be replaced by

$$\partial^2 W/\partial x^2 = (\partial^2 W/\partial \bar{r}^2)\cos^2\theta - 2(\partial^2 W/\partial \bar{r}\,\partial\theta)\sin\theta\cos\theta/\bar{r} + (\partial W/\partial \bar{r})\sin^2\theta/\bar{r} + 2(\partial W/\partial\theta)\sin\theta\cos\theta/\bar{r}^2 + (\partial^2 W/\partial\theta^2)\sin^2\theta/\bar{r}^2, \tag{2a}$$

$$\partial^2 W/\partial y^2 = (\partial^2 W/\partial \bar{r}^2)\sin^2\theta + 2(\partial^2 W/\partial\theta\,\partial \bar{r})\sin\theta\cos\theta/\bar{r} + (\partial W/\partial \bar{r})\cos^2\theta/\bar{r} - 2(\partial W/\partial\theta)\sin\theta\cos\theta/\bar{r}^2 + (\partial^2 W/\partial\theta^2)\cos^2\theta/\bar{r}^2, \tag{2b}$$

$$\partial^2 W/\partial x\,\partial y = (\partial^2 W/\partial \bar{r}^2)\cos\theta\sin\theta - (\partial^2 W/\partial\theta^2)\sin\theta\cos\theta/\bar{r}^2 - (\partial W/\partial \bar{r})\cos\theta\sin\theta/\bar{r} + (\partial^2 W/\partial \bar{r}\,\partial\theta)((\cos^2\theta - \sin^2\theta)/\bar{r}) + (\partial W/\partial\theta)((\sin^2\theta - \cos^2\theta)/\bar{r}^2). \tag{2c}$$

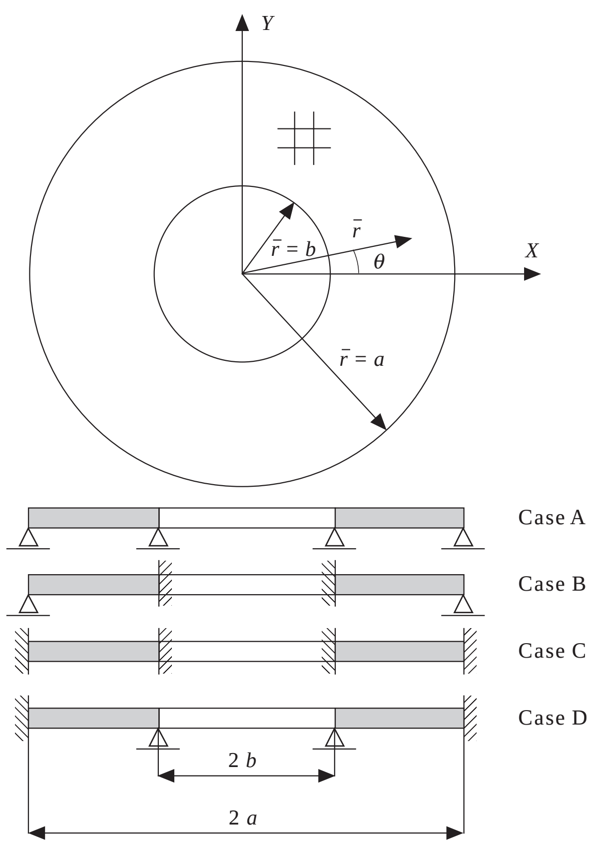

Figure 1. Rectangularly orthotropic, circular, annular plates executing transverse vibrations considered in the present investigation.

TABLE 1

Values of Ω_1 for the configuration simply supported at both edges; Case A

		Values of $\Omega_1 = \sqrt{\rho h/D_1}\omega_1 a^2$								
		$b/a = \frac{1}{4}$			$b/a = \frac{1}{2}$			$b/a = \frac{3}{4}$		
μ_2	D_k/D_1	$D_2/D_1 = \frac{1}{2}$	$D_2/D_1 = 1$	$D_2/D_1 = 2$	1/2	1	2	1/2	1	2
1/6	1/6	15·43†	17·400	20·800	33·170	37·420	44·730	131·310	148·140	177·060
		15·442‡	17·421	20·823	33·380	37·660	45·010	132·972	150·013	179·307
		14·927§	17·421	20·038	33·113	37·660	41·744	118·260	150·013	159·062
		14·896‖	17·343	19·909	30·846	37·396	41·312	115·737	146·196	156·648
1/3	1/3	16·770	18·610	21·820	36·060	40·010	46·910	142·750	158·370	185·700
		16·788	18·624	21·839	36·291	40·261	47·210	144·534	160·370	188·052
		16·336	18·624	21·110	34·068	40·261	44·001	128·739	160·370	167·245
		16·279	18·611	21·067	33·317	40·011	43·623	123·402	158·264	163·677
2/3	2/3	19·180	20·800	23·720	41·240	44·730	51·000	163·24	177·060	201·880
		19·198	20·823	23·742	41·499	45·013	51·323	165·306	179·314	204·436
		18·837	20·823	23·103	39·266	45·013	48·180	146·294	175·853	182·062
		18·4337	20·651	23·021	38·317	44·661	47·663	134·976	172·761	174·515
1/6	2/3	18·360	20·070	23·110	39·160	42·780	49·260	153·520	168·150	194·120
		18·395	20·106	23·139	39·306	43·011	49·589	155·646	170·425	196·705
		17·927	20·106	22·403	36·974	43·011	46·356	138·251	170·425	175·151
		17·752	20·038	22·351	36·296	42·663	45·883	129·574	166·729	169·565

† $W(r, \theta) \cong f(r)$ [4]. ‡ $W(r, \theta) \cong R(r)$. § $W(r, \theta) \cong R(r)\Theta(\theta)$; $[\gamma, \eta_1, \eta_2]$: optimization parameters. ‖ Finite element results.

TABLE 2

Values of Ω_1 for the configuration simply supported at $\bar{r} = a$ and clamped at $\bar{r} = b$; Case B

		Values of $\Omega_1 = \sqrt{\rho h/D_1}\omega_1 a^2$								
		$b/a = \frac{1}{4}$			$b/a = \frac{1}{2}$			$b/a = \frac{3}{4}$		
μ_2	D_k/D_1	$D_2/D_1 = \frac{1}{2}$	$D_2/D_1 = 1$	$D_2/D_1 = 2$	1/2	1	2	1/2	1	2
1/6	1/6	21·63†	24·410	29·170	49·67†	56·040	66·980	202·380	228·320	272·890
		21·59‡	24·360	29·110	49·69‡	56·060	67·010	202·130	228·040	272·560
		20·198§	24·360	27·117	44·763§	56·060	60·203	177·646	228·040	239·330
		–	–	–	44·560‖	55·695	60·007	–	–	–
1/3	1/3	23·520	26·090	36·300	54·000	59·910	70·250	220·010	244·080	286·210
		23·470	26·040	30·540	54·020	59·930	70·280	219·740	243·780	285·860
		22·093	26·040	28·564	48·675	59·930	63·277	192·682	243·781	251·220
		–	–	–	47·860	59·896	63·018	–	–	–
2/3	2/3	26·900	29·17	33·260	61·750	66·980	76·370	251·590	272·890	311·140
		26·840	29·11	33·200	61·783	67·013	76·407	251·288	272·560	310·760
		25·422	29·111	31·245	55·196	67·013	68·837	217·259	272·562	272·490
		–	–	–	52·593	66·534	67·684	–	–	–
1/6	2/3	24·660	27·130	31·490	57·170	62·780	72·720	234·760	257·450	297·700
		24·610	27·070	31·430	57·190	62·813	72·750	234·480	257·150	297·340
		23·151	27·070	29·417	51·159	62·813	65·393	203·913	257·150	260·834
		–	–	–	49·432	62·538	54·772	–	–	–

† $W(r,\theta) \cong f(r)$ [4]. ‡ $W(r,\theta) \cong R(r)$. § $W(r,\theta) \cong R(r)\Theta(\theta)$; $[\gamma, \eta_1, \eta_2]$: optimization parameters. ‖ Finite element results.

TABLE 3

Values of Ω_1 for the configuration clamped at both edges; Case C

		Values of $\Omega_1 = \sqrt{\rho h/D_1}\omega_1 a^2$								
		$b/a = \frac{1}{4}$			$b/a = \frac{1}{2}$			$b/a = \frac{3}{4}$		
μ_2	D_k/D_1	$D_2/D_1 = \frac{1}{2}$	$D_2/D_1 = 1$	$D_2/D_1 = 2$	1/2	1	2	1/2	1	2
1/6	1/6	32·76†	36·96	44·17	74·05	83·540	99·85	298·600	336·87	402·640
		32·70‡	36·898	44·101	74·006	83·490	99·789	296·714	334·739	400·089
		30·337§	36·898	40·721	66·146	83·490	88·958	260·560	334·739	350·820
		30·296‖	36·834	40·656	65·786	82·823	88·805	256·555	322·406	348·640
1/3	1/3	35·61	39·510	46·33	80·50	89·310	104·73	324·620	360·13	422·290
		35·556	39·445	46·254	80·453	89·254	104·660	332·562	357·851	419·617
		33·191	39·445	42·904	72·029	89·254	93·545	283·012	357·851	368·487
		33·038	39·443	42·846	70·553	89·523	93·037	267·334	357·797	362·826
2/3	2/3	40·72	44·170	50·36	92·06	99·850	113·85	371·210	402·64	459·080
		40·659	44·101	50·283	92·001	99·789	113·778	368·864	400·089	456·170
		38·180	44·101	46·939	81·910	99·789	101·860	320·197	400·089	400·230
		37·581	43·952	46·723	77·961	98·251	99·778	283·899	376·905	378·064
1/6	2/3	38·25	41·900	48·39	86·47	94·730	109·38	348·690	381·98	441·070
		38·193	41·838	48·311	86·420	94·669	109·314	346·488	379·558	438·276
		35·790	41·838	44·972	77·243	94·669	97·830	302·698	379·558	384·912
		35·434	41·793	44·858	74·484	94·208	96·613	276·012	369·749	371·136

† $W(r, \theta) \cong f(r)$ [4]. ‡ $W(r, \theta) \cong R(r)$. § $W(r, \theta) \cong R(r)\Theta(\theta)$; $[\gamma, \eta_1, \eta_2]$: optimization parameters. ‖ Finite element results.

TABLE 4

Values of Ω_1 for the configuration clamped at the outer boundary and simply supported at the inner contour; Case D

		Values of $\Omega_1 = \sqrt{\rho h/D_1}\omega_1 a^2$								
		$b/a = \frac{1}{4}$			$b/a = \frac{1}{2}$			$b/a = \frac{3}{4}$		
μ_2	D_k/D_1	$D_2/D_1 = \frac{1}{2}$	$D_2/D_1 = 1$	$D_2/D_1 = 2$	1/2	1	2	1/2	1	2
1/6	1/6	24·65†	27·81	33·23	53·02†	59·81	71·49	207·76	234·39	280·13
		24·702‡	27·868	33·309	52·97‡	59·758	71·42	207·185	234·271	279·369
		23·753§	27·868	31·868	48·635§	59·758	65·223	183·689	234·271	246·815
		–	–	–	48·438‖	59·679	54·956	–	–	–
1/3	1/3	26·79	29·73	34·86	57·64	63·94	74·98	225·85	250·55	293·81
		26·854	29·792	34·934	57·58	64·884	74·911	225·234	250·449	293·004
		26·018	29·7922	33·592	53·245	63·884	68·764	200·056	250·449	259·771
		–	–	–	52·868	63·860	68·503	–	–	–
2/3	2/3	30·64	33·23	37·89	65·91	71·49	81·51	258·25	280·11	319·39
		30·709	33·309	37·978	65·851	71·425	81·437	258·110	279·974	319·213
		30·360	33·309	36·796	61·319	71·425	75·300	227·628	279·974	282·923
		–	–	–	60·169	71·265	74·766	–	–	–
1/6	2/3	29·86	32·53	37·30	63·03	68·84	79·20	244·41	267·42	308·31
		29·92	32·600	37·376	63·024	68·834	79·182	244·254	267·244	308·118
		29·116	32·600	36·074	58·598	68·834	73·69	216·532	267·244	273·291
		–	–	–	57·8651	68·759	72·700	–	–	–

† $W(r, \theta) \cong f(r)$ [4]. ‡ $W(r, \theta) \cong R(r)$. § $W(r, \theta) \cong R(r)\Theta(\theta)$; $[\gamma, \eta_1, \eta_2]$: optimization parameters. ‖ Finite element results.

The previous investigation [4] assumed that the fundamental mode shape can be approximated by

$$W(r, \theta) \simeq W_a(r), \qquad r = \bar{r}/a. \tag{3}$$

Clearly, this approximation does not take into account the azimuthal variations introduced by the constitutive properties of the orthotropic material and in this respect, the fundamental eigenvalues determined in reference [4] must be considered as first order approximations.

The approximation (3) can be improved if one makes

$$W(r, \theta) \simeq W_a(r, \theta) = R(r)\Theta(\theta), \tag{4}$$

where $R(r)$ is a functional relation which satisfies, at least, the essential boundary conditions of the mechanical system. The function $R(r)$ will contain an exponential parameter γ which will allow for minimization of the fundamental frequency coefficient $\Omega_1 = \sqrt{\rho h/D_1}\omega_1 a^2$ [10]. After several numerical experiments it was decided to express $\Theta(\theta)$ as

$$\Theta(\theta) = 1 + \eta_1 \sin^2\theta + \eta_2 \cos^2\theta, \tag{5}$$

where η_1 and η_2 are, also, optimization parameters.

It is important to point out that the open literature presents very few cases where the approximating function takes into account the azimuthal dependence, the exception being the analysis presented in reference [11]. Based on previous studies [10], the following expressions were chosen for $R(r)$.

Case A:

$$R(r) = A_1(1 - r^\gamma)[1 - (r/b_1)^2] + A_2(1 - r^{\gamma+1})[1 - (r/b_1)^2]; \qquad b_1 = b/a. \tag{6}$$

Case B:

$$R(r) = A_1(1 - r^\gamma)(1 - r/b_1)^2 + A_2(1 - r^{\gamma+1})(1 - r/b_1)^2. \tag{7}$$

Case C:

$$R(r) = A_1(1 - r^\gamma)^2[1 - (r/b_1)]^2 + A_2(1 - r^{\gamma+1})^2[1 - (r/b_1)]^2. \tag{8}$$

Case D:

$$R(r) = A_1(1 - r^\gamma)^2[1 - (r/b_1)^2] + A_2(1 - r^{\gamma+1})^2[1 - (r/b_1)]^2. \tag{9}$$

3. NUMERICAL RESULTS AND CONCLUSIONS

Tables 1, 2, 3 and 4 present values of the fundamental frequency coefficients Ω_1 for Cases A, B, C and D. The values of Ω_1 are tabulated as a function of D_2/D_1, D_k/D_1 and μ_2 according to the combinations of orthotropic parameters chosen in reference [4]. In all cases the Tables contain, for each particular configuration: (1) the value calculated in [4], (2) the value determined in the present investigation using $R(r)$, (3) the value calculated in the present study using $R(r)\Theta(\theta)$ and, in some instances, (4) the eigenvalue obtained by means of the finite element method [8].

The number of elements used was varied as a function of the parameter b/a. For $b/a = 1/4$, 6424 elements were used; for $b/a = 1/2$, 6380 elements and for $b/a = 3/4$, 6408 elements.‡ From the analysis of Table 1 one concludes that the values obtained in reference [4] are slightly lower than those obtained in the present study when taking $W \simeq R(r)$. This is due to the fact that the condition of nulle moment normal to the edges was approximately satisfied in reference [4]. Nevertheless, the results obtained making $W \simeq R(r)\Theta(\theta)$ are always lower than those available in reference [4] and also are in good agreement with the finite element values.

In the case of Table 2, Case b, the results obtained using $R(r)$ are in some instances lower and, hence, more accurate than those obtained in reference [4]. In general the conclusions are similar to those drawn in the case of Table 1.

Examining now Table 3, one concludes that now the values obtained using $R(r)$ are, in general, lower than those determined in reference [4]. The agreement with the finite element values is quite satisfactory for most of the situations.

In the case of Table 4 (depicting values of Ω_1 for Case D) one can draw the same conclusions as in the case of Table 1; the results obtained in reference [4] are, in general, more accurate than those obtained in the present investigation when making $W \simeq R(r)$.

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‡ The number of resulting equations were 19432, 19430 and 19437, respectively.

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